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A NUMERICAL MODEL FOR NON-LINEAR WAVE DIFFRACTION AROUND LARGE OFFSHORE STRUCTURES

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SUMMARY

The present paper describes a numerical model for predicting the non-linear wave forces exerted on large coastal or offshore structures. The interaction is calculated from a known initial condition corresponding to still water in the immediate vicinity of the structure. A prescribed incident wave of large amplitude approaches the structure. The boundary constraints on the free surface are considered in the fully non-linear versions. Not far away from the structure, an artificial open boundary is considered with a suitable description of the radiation boundary condition for an incident wave propagating inwards, and in addition the scattering wave being absorbed. The finite difference method and time stepping are adopted for numerical calculation in the present model. For illustration the wave forces on a surface-piercing vertical cylinder subjected to two different incident wave trains were evaluated. Reasonably good agreement could be obtained between the numerical results and the analytical solution given by Isaacson.[1]

KEY WORDS Non-linear Wave Diffraction Offshore Structure Artificial Open Boundary Radiation Condition FDM

INTRODUCTION

Based on linearized theory, the wave diffraction problem has been studied extensively. In this case the non-linear boundary conditions over the free surface are linearized, and in addition these constraints are applied approximately on the still water level in accordance with the small amplitude assumption, together with other boundary conditions over rigid walls, as well as the Sommerfeld radiation condition at infinity; the velocity potential of the scattered wave is solved from the governing equation. Within the framework of linear problems, the physical model is well posed, and the governing equation can be solved readily by the separation of variables method. The solution domain bounded by a time-dependent free surface is approximated by a fixed spatial domain. The difficulties encountered in the solution process are thus reduced to a great extent.

For the non-linear wave diffraction problem, the methods which have been applied successfully in the linear case are no longer available. The difficulties are primarily associated with the non-linear boundary constraints on the free surface, wherein its position varies with time, and is unknown *a priori.* These conditions cannot be written so as to be satisfied approximately on the still water level because of the steep wave nature. In addition, the variables associated with the total wave must be considered instead of those of the diffraction wave; another difficulty concerning a suitable description of the radiation condition is then set up. In view of these facts,

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a realistic computational model for non-linear wave diffraction is of great importance. Moreover, because of the non-linearity, the separation of variables method is invalid, and also the free surface cannot be approximated by the still water level, so that the time-variant boundary must be treated specially. The difficulties associated with the non-linear problem are thus enhanced.

A method used to calculate the non-linear diffraction is applicable based on Stokes expansion,[2–4] in which the total wave potential is composed of a finite series of velocity potentials of different orders in the sense of a small parameter expansion. The sequential calculation of different order potentials is similar to that of the linear problem. It cannot therefore have widespread applications, owing to its relatively restricted conditions.

An alternative approach to the non-linear diffraction problem, in which the total velocity potential is taken as the variable of consideration, has been adopted by Isaacson.[1] The wave diffraction is treated as a transient problem, with a given wave form approaching a structure, within the immediate vicinity of which an initial condition corresponding to still water is prescribed. Over a control surface lying a sufficient distance away, the boundary condition is given as a known incident wave potential. Calculations can be advanced until the scattered wave approaches this control surface.

The objective of the present paper is to develop a new computational model which would be capable of analysing numerically the non-linear diffraction in the general case over a sufficient duration to describe the wave evolution with time. In this case, the key to success lies in the treatment of boundary constraints over the free surface and in the elimination of reflection or wrap-around from the outer boundary of the spatially discrete grids. In view of reducing the requirement on computer storage and computing cost, the location of the outer boundary should not be far away from the object. For this, a suitable radiation boundary condition instead of that of Isaacson's model is prescribed on an artificial open boundary lying a moderate distance from the structure. Over the open boundary, the incident wave is allowed to propagate through; also, the scattered wave can propagate outwards, and no wave reflection results there. Improper treatment of the open boundary condition would result in spurious reflections that would make the computation distorted from the real physical phenomenon. Provided that the open boundary is a moderate distance from the structure, the scattered wave which results from the interaction of the approaching wave with the structure would be expected to retain the character of plane parallelism in the vicinity of a spatial-temporal node M on this boundary. Then, the physical variable ϕ_s (say, velocity potential, or wave height) of the scattered wave is propagating outwards with celerity C in the direction of $\mathbf{l}$. Suppose that the open boundary lies a sufficient distance away, so that the body can be imagined as a point relative to the domain bounded by the open boundary, then the direction of $\mathbf{l}$ should coincide with the radiated line from the centre. In order to reduce the computational effort, it is preferable to have the open boundary lying a moderate distance away, where the scattered wave can be considered plane parallel within a grid spacing, but the direction of $\mathbf{l}$ becomes unknown. Fortunately, in view of the nature of plane parallelism, the propagating direction $\mathbf{l}$ can be formulated with the variables of the wave itself. Cheng[5] used a similar thought in the problem of wave radiation in the absence of an incident wave. For the problem of wave diffraction, the variables of the total wave are considered in the present model. The variables of the scattered wave is substituted by $\phi_s = \phi - \phi_I$, where ϕ and ϕ_I denote the variables corresponding to the total wave and incident wave, respectively.

In the present model, the diffraction generated by sinusoidal and solitary waves approaching and propagating over a surface-piercing vertical cylinder is described. The governing equation, incorporating the boundary conditions, is solved by means of a finite difference method, and a time-stepping procedure is also used to obtain the wave evolution with time. Computed results of wave force corresponding to different locations of the open boundary are presented. In particular, preliminary comparisons are made between the numerical results and the closed-form solution given by Isaacson, and these are quite favourable. They show that the present model is capable of predicting non-linear wave forces, and that the treatment of the open boundary condition is effective in absorbing the inward reflection of the scattered waves.

FORMULATION OF THE PROBLEM

The solution domain of interest (Figure 1) is bounded by an inner surface S_b, the sea-bed S_{bt}, the free surface S_f and an open boundary S_c, where S_b identifies the immersed surface of the cylinder with radius R_1, and S_c a cylindrical surface with radius R_2. Let r, θ, z form a cylindrical co-ordinate system, with z measured upwards from the still water level. The fluid is assumed inviscid and incompressible, and the flow is irrotational. The fluid motion can therefore be described by a velocity potential ϕ, which satisfies the Laplace equation

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}=0, \tag{1}$$

and is subject to the following boundary conditions:

$$\partial\phi/\partial r=0, \quad \text{on } S_b\ (r=R_1), \tag{2a}$$

$$\partial\phi/\partial z=0, \quad \text{on } S_{bt}\ (z=-d), \tag{2b}$$

and the dynamic and kinetic conditions on S_f:

$$\frac{\partial\eta}{\partial t}+\frac{\partial\phi}{\partial r}\frac{\partial\eta}{\partial r}+\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}\frac{\partial\eta}{\partial\theta}-\frac{\partial\phi}{\partial z}=0, \quad \text{on } S_f\ (z=\eta), \tag{3}$$

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}+\frac{\eta}{Fr^2}=0, \quad \text{on } S_f\ (z=\eta), \tag{4}$$

together with a suitable radiation boundary condition posed on the open boundary, which will be discussed in detail later. In these formulae, η denotes the wave elevation, d the water depth, Fr denotes the Froude number $U/(gL)^{1/2}$ and g is the gravity acceleration. U and L are the characteristic velocity and length, respectively.

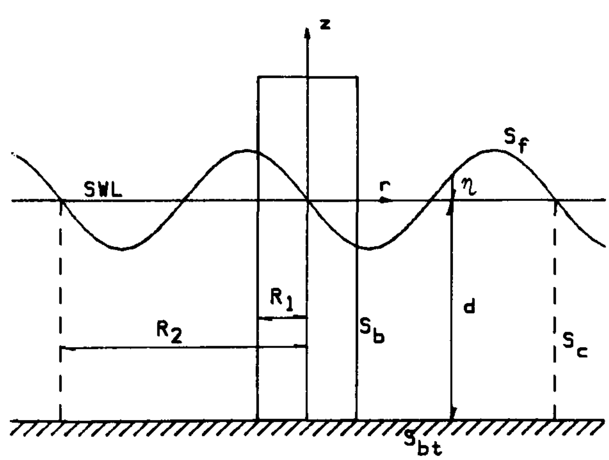

Figure 1

Initial conditions are chosen to correspond to still water in the vicinity of the structure and a prescribed steep incident wave form approaching inwards.

OPEN BOUNDARY CONDITION

An artificial open boundary is considered lying a moderate distance away from the structure (Figure 2). The variable of the scattered wave ϕ_s (say, velocity potential, or wave elevation) in the immediate vicinity of a spatial-temporal node M (r_m, θ_m, z_m, t) over the open boundary should have the form

$$\phi_s = \phi_s(l - Ct), \tag{5}$$

or equivalently

$$\phi_s = \phi_s(K_r r + K_\theta R_2 \theta - Ct). \tag{6}$$

In physical terms, this implies that within the immediate vicinity of the node M the scattered wave can be approximated as a plane parallel wave, propagating outwards with the celerity C along the direction of $\mathbf{l}$, and the derivatives of $\mathbf{K}_r$, $\mathbf{K}_\theta$ and C with respect to r, θ and t are relatively small and thus can be ignored, so that from equations (5) and (6), the following equations are straightforwardly derived:

$$\partial\phi_s/\partial t + C\,\partial\phi_s/\partial l = 0, \tag{7}$$

or

$$\partial\phi_s/\partial t + C(K_r\,\partial\phi_s/\partial r + K_\theta\,\partial\phi_s/R_2\,\partial\theta) = 0, \tag{8}$$

and

$$K_r^2 + K_\theta^2 = 1. \tag{9}$$

Comparison of equations (5) and (6) shows that

$$K_r = \cos(\mathbf{r}, \mathbf{l}), \quad K_\theta = \cos(\theta, \mathbf{l}),$$

i.e. K_r and K_θ can be considered as direction cosines of $\mathbf{l}$ with respect to r and θ, respectively.

If the open boundary is not placed at a sufficient distance, $\mathbf{l}$ will in general not be parallel with $\mathbf{r}$. The direction of $\mathbf{l}$ is, or the direction cosines K_r and K_θ are, then unknown. However, provided that the open boundary lies a moderate distance from the body, where the scattered wave can be considered as a plane parallel wave within a computing grid, the direction of $\mathbf{l}$ can

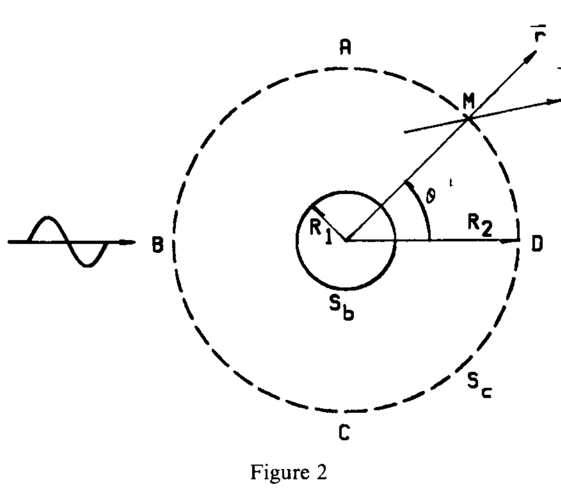

Figure 2

then be evaluated from the variables of the wave itself. Differentiating partially the terms of the two sides of equation (6) with respect to r and θ results in

$$\partial\phi_s/\partial r = K_r\,\partial\phi_s/\partial a \tag{10}$$

$$\partial\phi_s/R_2\,\partial\theta = K_\theta\,\partial\phi_s/\partial a, \tag{11}$$

where $a = K_r r + K_\theta R_2\theta - Ct$. Squaring the terms of equations (10) and (11), and adding them together, gives the following expression pertaining to equation (9):

$$(\partial\phi_s/\partial a)^2 = (\partial\phi_s/\partial r)^2 + (\partial\phi_s/R_2\,\partial\theta)^2, \tag{12}$$

or

$$\partial\phi_s/\partial a = \pm[(\partial\phi_s/\partial r)^2 + (\partial\phi_s/R_2\,\partial\theta)^2]^{1/2}. \tag{13}$$

Substituting equation (13) into equations (10) and (11), we obtain

$$K_r = \alpha(\partial\phi_s/\partial r)[(\partial\phi_s/\partial r)^2 + (\partial\phi_s/R_2\,\partial\theta)^2]^{-1/2}, \tag{14}$$

$$K_\theta = \alpha(\partial\phi_s/R_2\,\partial\theta)[(\partial\phi_s/\partial r)^2 + (\partial\phi_s/R_2\,\partial\theta)^2]^{-1/2}. \tag{15}$$

With equations (14) and (15) being substituted into equations (3) and (4), we obtain

$$\frac{\partial\phi_s}{\partial t} + \alpha C\left\{\frac{\partial\phi_s/\partial r}{\left[\left(\frac{\partial\phi_s}{\partial r}\right)^2 + \left(\frac{\partial\phi_s}{R_2\,\partial\theta}\right)^2\right]^{1/2}}\frac{\partial\phi_s}{\partial r} + \frac{\partial\phi_s/R_2\,\partial\theta}{\left[\left(\frac{\partial\phi_s}{\partial r}\right)^2 + \left(\frac{\partial\phi_s}{R_2\,\partial\theta}\right)^2\right]^{1/2}}\frac{\partial\phi_s}{R_2\,\partial\theta}\right\} = 0, \tag{16}$$

where the coefficient α is defined as

$$\alpha = \begin{cases} 1, & \text{if } \partial\phi_s/\partial r > 0, \\ -1, & \text{if } \partial\phi_s/\partial r < 0. \end{cases} \tag{17}$$

From equation (14), K_r should have the same sign as $\alpha\,\partial\phi_s/\partial r$, where $K_r = \cos(\mathbf{r}, \mathbf{l})$, and also since the scattered wave ϕ_s corresponds to an outward travelling wave, it requires that the absolute value of angle between $\mathbf{r}$ and $\mathbf{l}$ be smaller than $\pi/2$, so that $K_r > 0$; this implies that $\alpha\,\partial\phi_s/\partial r > 0$, so that equation (17) then holds true.

Equation (16) can be applied as the radiation boundary condition on the open boundary, where C is the local celerity. Let

$$\phi_s = \phi - \phi_I, \tag{18}$$

where ϕ and ϕ_I denote the variables corresponding to total wave and incident wave, respectively; we then have the solution condition in terms of the total wave over the open boundary.

Let ϕ denote the velocity potential and η the wave elevation, then the radiation conditions over the open boundary can be written as

$$\frac{\partial(\phi-\phi_I)}{\partial t} + \alpha C\left\{\frac{\dfrac{\partial(\phi-\phi_I)}{\partial r}\dfrac{\partial(\phi-\phi_I)}{\partial r}}{\sqrt{\left[\left(\dfrac{\partial(\phi-\phi_I)}{\partial r}\right)^2 + \left(\dfrac{\partial(\phi-\phi_I)}{R_2\,\partial\theta}\right)^2\right]}} + \frac{\dfrac{\partial(\phi-\phi_I)}{R_2\,\partial\theta}\dfrac{\partial(\phi-\phi_I)}{R_2\,\partial\theta}}{\sqrt{\left[\left(\dfrac{\partial(\phi-\phi_I)}{\partial r}\right)^2 + \left(\dfrac{\partial(\phi-\phi_I)}{R_2\,\partial\theta}\right)^2\right]}}\right\} = 0, \tag{19}$$

$$\frac{\partial(\eta-\eta_I)}{\partial t}+\alpha C\left\{\frac{\dfrac{\partial(\eta-\eta_I)}{\partial r}\dfrac{\partial(\eta-\eta_I)}{\partial r}}{\sqrt{\left[\left(\dfrac{\partial(\eta-\eta_I)}{\partial r}\right)^2+\left(\dfrac{\partial(\eta-\eta_I)}{R_2\,\partial\theta}\right)^2\right]}}+\frac{\dfrac{\partial(\eta-\eta_I)}{R_2\,\partial\theta}\dfrac{\partial(\eta-\eta_I)}{R_2\,\partial\theta}}{\sqrt{\left[\left(\dfrac{\partial(\eta-\eta_I)}{\partial r}\right)^2+\left(\dfrac{\partial(\eta-\eta_I)}{R_2\,\partial\theta}\right)^2\right]}}\right\}=0, \tag{20}$$

and equation (19) can be rewritten simply in terms of velocity components as

$$\frac{\partial(\phi-\phi_I)}{\partial t}+\alpha C\left\{\frac{(u-u_I)(u-u_I)}{\sqrt{[(u-u_I)^2+(v-v_I)^2]}}+\frac{(v-v_I)(v-v_I)}{\sqrt{[(u-u_I)^2+(v-v_I)^2]}}\right\}=0. \tag{21}$$

Much simpler alternatives for the terms in the braces in equations (19)–(21) can be adopted, but we prefer to take the outlined forms, in which the direction cosines of **l** remain explicitly in the equation. In practical problems, over the open boundary, lying a moderate distance from the structure, the direction cosines must be independent of the variable z, or at nodes of different heights waves will propagate along the same direction.

Errors are not excluded in the numerical procedure; sometimes there will be divergence. As the direction cosines were kept in an explicit form in the equation, it should be possible to apply some smoothing process in calculating the direction cosines along the vertical axis, so as to improve stability of computation.

CO-ORDINATE TRANSFORMATION

The governing equation (1) in terms of the velocity potential $\phi(r,\theta,z,t)$ is defined over a time-varying spatial domain, i.e. the flow field is bounded partly by the free surface, whose location is unknown *a priori*. In order to avoid the trouble arising from the time-dependence of the grid in space discretization, the following co-ordinate transformation is introduced:

$$\begin{aligned} x&=r,\\ y&=\theta,\\ \xi&=\frac{z+d}{\eta+d}. \end{aligned} \tag{22}$$

The transformed flow field becomes a time-invariant domain of solution, with the free surface mapped onto a plane $\xi=1$. In the x, y, ξ co-ordinate system the primitive Laplace equation (1) is transformed into the following equation with variable coefficients:

$$C_1\frac{\partial^2\phi}{\partial x^2}+C_2\frac{\partial^2\phi}{\partial y^2}+C_3\frac{\partial^2\phi}{\partial x\,\partial\xi}+C_4\frac{\partial^2\phi}{\partial y\,\partial\xi}+C_5\frac{\partial^2\phi}{\partial\xi^2}+C_6\frac{\partial\phi}{\partial x}+C_7\frac{\partial\phi}{\partial\xi}=0, \tag{23}$$

where

$$C_1=1,\quad C_2=1/x^2,\quad C_3=-\frac{2\xi\,\partial\eta/\partial x}{\eta+d},\quad C_4=-\frac{2\xi\,\partial\eta/\partial y}{(\eta+d)x^2},$$

$$C_5=\frac{1}{(\eta+d)^2}\left\{1+\xi^2\left[\left(\frac{\partial\eta}{\partial x}\right)^2+\left(\frac{\partial\eta}{x\,\partial y}\right)^2\right]\right\},\quad C_6=1/x,$$

$$C_7 = \frac{2\xi\left[\left(\frac{\partial\eta}{\partial x}\right)^2 + \left(\frac{\partial\eta}{x\,\partial y}\right)^2\right]}{(\eta+d)^2} - \frac{\xi\left(\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{x^2\,\partial y^2}\right)}{\eta+d} - \frac{\xi\frac{\partial\eta}{\partial x}}{x(\eta+d)}.$$

The corresponding boundary conditions (2a) and (2b), after transformation become

$$\frac{\partial\phi}{\partial x} = \frac{\xi\,\partial\eta/\partial x}{\eta+d}\frac{\partial\phi}{\partial\xi}, \quad \text{on } S_b(x=R_1), \tag{24}$$

$$\frac{\partial\phi}{\partial\xi} = 0, \quad \text{on } S_{bt}(\xi=0), \tag{25}$$

and the corresponding boundary conditions (3), (4), (20) and (21) will keep the primitive forms, except that the variables r, θ, z should be substituted by x, y, ξ, and also the involved velocity components u, v, w or $\partial\phi/\partial r$, $\partial\phi/r\,\partial\theta$ and $\partial\phi/\partial z$ must be evaluated in the new co-ordinate system as

$$u = \frac{\partial\phi}{\partial r} = \frac{\partial\phi}{\partial x} - \frac{\xi\frac{\partial\eta}{\partial x}}{\eta+d}\frac{\partial\phi}{\partial\xi}, \tag{26}$$

$$v = \frac{\partial\phi}{r\,\partial\theta} = \frac{\partial\phi}{x\,\partial y} - \frac{\xi\frac{\partial\eta}{x\,\partial y}}{\eta+d}\frac{\partial\phi}{\partial\xi}, \tag{27}$$

$$w = \frac{\partial\phi}{\partial z} = \frac{1}{\eta+d}\frac{\partial\phi}{\partial\xi}. \tag{28}$$

For the sake of simplicity, these boundary conditions are now rewritten in abbreviated formulae:

$$\frac{\partial\eta}{\partial t} = F_1, \quad \text{on } S_f(\xi=1), \tag{29}$$

$$\frac{\partial\phi}{\partial t} = F_2, \quad \text{on } S_f(\xi=1), \tag{30}$$

$$\frac{\partial\eta}{\partial t} = F_3, \quad \text{on } S_c(x=R_2). \tag{31}$$

$$\frac{\partial\phi}{\partial t} = F_4, \quad \text{on } S_c(x=R_2). \tag{32}$$

This completes an outline of the theoretical basis of the non-linear wave diffraction problems concerned.

NUMERICAL PROCEDURE

In a numerical procedure for solving the initial-boundary value problem outlined, the transformed domain of computation is discretized into a finite number of cylindrical cells with grid spacings of Δx, Δy and $\Delta\xi$ parallel to the x, y and ξ axes, respectively. The finite difference method is used. Whenever the velocity potential ϕ over the domain, the wave elevation η and the velocity components u, v and w are known at time t, the values of these variables at time $t+\Delta t$ are evaluated according to the following process: update the wave elevation and velocity potential over both the free surface S_f and the open boundary S_c at time $t+\Delta t$; solve a finite difference equation iteratively to obtain the distribution of ϕ over the entire domain; modify the wave elevation and ϕ over S_f+S_c; re-iterate the solution of ϕ over the domain in accordance with the modified boundary values of ϕ and η. Each step of the numerical procedure is outlined separately below.

Updating the wave elevation and velocity potential over S_f+S_c

The free surface boundary conditions and radiation condition over S_c are the only ones to involve the time variable. A time-stepping procedure is adopted so that the initial value problem can be posed at any instant in terms of known quantities relating to previous time steps. Derivatives with respect to time are approximated by forward differences. The difference approximations of equations (29)–(32) are

$$\eta(t+\Delta t)=\eta(t)+\Delta t F_1(t), \quad \text{on } S_f(\xi=1), \tag{33}$$

$$\phi(t+\Delta t)=\phi(t)+\Delta t F_2(t), \quad \text{on } S_f(\xi=1), \tag{34}$$

$$\eta(t+\Delta t)=\eta(t)+\Delta t F_3(t), \quad \text{on } S_c(x=R_2), \tag{35}$$

$$\phi(t+\Delta t)=\phi(t)+\Delta t F_4(t), \quad \text{on } S_c(x=R_2). \tag{36}$$

Solving the difference approximation of the governing equation iteratively to obtain velocity potential over the domain

Once the wave elevation relating to time $t+\Delta t$ is known, the variable coefficients C_i in equation (23) are evaluated, and the velocity potential ϕ over S_f+S_c becomes known by equations (34) and (36), together with $\partial\phi/\partial n$ over S_b+S_{bt}, given by equations (24) and (25). This completes all the necessary boundary conditions for solving equation (23) relating to time $t+\Delta t$. The terms involved in equation (23) are approximated by central difference forms:

$$\frac{\partial\phi}{\partial x}=(\phi_{i+1,j,k}-\phi_{i-1,j,k})/2\Delta x,$$

$$\frac{\partial\phi}{\partial\xi}=(\phi_{i,j,k+1}-\phi_{i,j,k-1})/2\Delta\xi,$$

$$\frac{\partial^2\phi}{\partial x^2}=(\phi_{i+1,j,k}-2\phi_{i,j,k}+\phi_{i-1,j,k})/(\Delta x)^2,$$

$$\frac{\partial^2\phi}{\partial x\,\partial\xi}=(\phi_{i+1,j,k+1}-\phi_{i-1,j,k+1}-\phi_{i+1,j,k-1}+\phi_{i-1,j,k-1})/4\,\Delta x\,\Delta\xi,$$

$$\frac{\partial^2\phi}{\partial y\,\partial\xi}=(\phi_{i,j+1,k+1}-\phi_{i,j-1,k+1}-\phi_{i,j+1,k-1}+\phi_{i,j-1,k-1})/4\,\Delta y\,\Delta\xi,$$

$$\frac{\partial^2\phi}{\partial y^2}=(\phi_{i,j+1,k}-2\phi_{i,j,k}+\phi_{i,j-1,k})/(\Delta y)^2,$$

$$\frac{\partial^2\phi}{\partial\xi^2}=(\phi_{i,j,k+1}-2\phi_{i,j,k}+\phi_{i,j,k-1})/(\Delta\xi)^2.$$

The difference approximation corresponding to equation (23) may be written as

$$B_1\phi_{i-1,j,k+1}+B_2\phi_{i,j,k+1}+B_3\phi_{i+1,j,k+1}+B_4\phi_{i-1,j,k}+B_5\phi_{i,j,k}+B_6\phi_{i+1,j,k}$$
$$+B_7\phi_{i-1,j,k-1}+B_8\phi_{i,j,k-1}+B_9\phi_{i+1,j,k-1}+B_{10}\phi_{i,j-1,k+1}+B_{11}\phi_{i,j-1,k}$$
$$+B_{12}\phi_{i,j-1,k-1}+B_{13}\phi_{i,j+1,k+1}+B_{14}\phi_{i,j+1,k}+B_{15}\phi_{i,j+1,k-1}=0, \tag{37}$$

where the coefficients B_m are functions of the C_ns of the partial differential equation as well as Δx, Δy and $\Delta\xi$ only.

The difference equation (37) is then solved at the new time step by an iterative scheme as

$$B_2\phi^{(n+1)}_{i,j,k+1}+B_5\phi^{(n+1)}_{i,j,k}+B_8\phi^{(n+1)}_{i,j,k-1}=R(\phi^{(n)},\phi^{(n+1)}), \tag{38}$$

where the superscripts denote the step number of iteration. This iterative scheme is mono-directional implicit along the ξ axis. With the equation (38) written on all the nodes within the domain or on the boundary where the Neumann type boundary condition is prescribed, a set of algebraic equations is solved iteratively to obtain the velocity potential at time $t+\Delta t$ over the domain.

Modifying the wave height and velocity potential

Once the velocity potential relating to time $t+\Delta t$ over the domain is known, the velocity components u, v, w can then be calculated. An average process is applied to modify the wave elevation and velocity potential over S_f+S_c:

$$\eta(t+\Delta t)=\eta(t)+\frac{\Delta t}{2}[F_1(t)+F_1(t+\Delta t)], \quad \text{on } S_f(\xi=1), \tag{39}$$

$$\phi(t+\Delta t)=\phi(t)+\frac{\Delta t}{2}[F_2(t)+F_2(t+\Delta t)], \quad \text{on } S_f(\xi=1), \tag{40}$$

$$\eta(t+\Delta t)=\eta(t)+\frac{\Delta t}{2}[F_3(t)+F_3(t+\Delta t)], \quad \text{on } S_c(x=R_2), \tag{41}$$

$$\phi(t+\Delta t)=\phi(t)+\frac{\Delta t}{2}[F_4(t)+F_4(t+\Delta t)], \quad \text{on } S_c(x=R_2). \tag{42}$$

Recalculating velocity potential at $t+\Delta t$

The procedure is similar to that of the second step, incorporating the modified wave elevation and velocity potential (39)–(42) over the boundary S_f+S_c.

The above procedure of solution is iterated until the velocity potential within the domain converges to within a prescribed error bound. In practice, results of favourable accuracy have been obtained over two steps of iteration, which has been adopted previously by Lu[6] to solve a non-linear wave problem in a finite flow field.

WAVE FORCE

At any node the pressure may be determined by the application of the unsteady Bernoulli equation

$$p=-\rho\left(\frac{\partial\phi}{\partial t}+\frac{V^2}{2}+gz\right). \tag{43}$$

The wave load on the structure is of primary interest and it can be calculated by appropriate integration of the pressure. Thus the component of force along the direction of wave propagation is expressed as

$$F = \iint_{S_b} \rho\left(\frac{\partial \phi}{\partial t} + \frac{V^2}{2} + gz\right)\cos\theta\, \mathrm{d}s. \tag{44}$$

Equations (43) and (44) are written in the cylindrical co-ordinate system (r, θ, z). Considering the co-ordinate transform, the wave force is rewritten as

$$F = \iint_{S_b} \rho\left\{\frac{\partial \phi}{\partial t} + \frac{V^2}{2} + g[(\eta + d)\xi - d]\right\}\cos y(\eta + d)\, \mathrm{d}\tau,$$

where $\mathrm{d}\tau$ is an infinitesimal cylindrical area in the x, y, ξ co-ordinate system. The non-dimensional form of wave force is then expressed as

$$\frac{F}{\rho g H R_1 d} = \frac{1}{H R_1 d}\iint_{S_b}\left\{Fr^2\left(\frac{\partial \phi}{\partial t} + \frac{V^2}{2}\right) + [(\eta + d)\xi - d]\right\}\cos y(\eta + d)\, \mathrm{d}\tau, \tag{45}$$

where the term $\partial\phi/\partial t$ is approximated by the central difference

$$\left(\frac{\partial \phi}{\partial t}\right) = \frac{1}{2\Delta t}[\phi(t + \Delta t) - \phi(t - \Delta t)]. \tag{46}$$

Within each grid the integration of (45) is implemented approximately by multiplying the grid area by the averaged value of pressures at four nodes of the grid.

RESULTS AND ANALYSIS

A computer program which incorporates the present model has been used to generate numerical results to verify the practical viability of this method. The program has been exemplified with two incident waves of steeper crest and small amplitude, respectively, interacting with a surface-piercing vertical cylinder. In the first case, a sinusoidal wave train of small amplitude is considered as the incident wave. The free surface elevation is expressed as

$$\eta = \frac{H}{2}\cos[k(x - ct)],$$

where H is the wave height, k is the wave number and c is the wave celerity. In addition, an intermediate wave profile precedent to the sinusoidal wave train is necessary, by which the prescribed incident wave can be connected smoothly with the still water level in the vicinity of the structure at the initial stage. This intermediate wave profile could be selected as a portion of a wave group or the like. The numerical results are reproduced in Figure 3, where the solid line indicates the analytical solution of MacCamy and Fuchs,[7] and the points marked by '▲' denote the numerical results by the present model.

In the second example, the incident wave under consideration was a solitary wave form given by

$$\eta = H\,\mathrm{sech}^2[\kappa(x_s - ct)], \tag{47}$$

where $\kappa = (3H/4d^3)^{1/2}$, $x_s = x + R_1 + 3{\cdot}0/\kappa$, together with water depth $d = 0{\cdot}5$, wave elevation $H = 0{\cdot}05$ and radius of cylinder $R_1 = 1$. Several results corresponding to different distances between the cylinder and the open boundary are generated to investigate numerically the effect

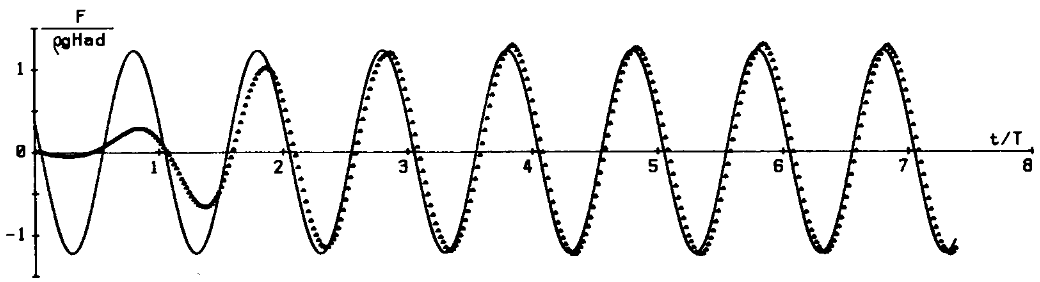

Figure 3. Wave force on a surface-piercing cylinder subjected to a incident sinusoidal wave train, with $d/R_1 = 2$, $H/R_1 = 0{\cdot}1$, $ka = 0{\cdot}75$, $R_2 = 2{\cdot}5$

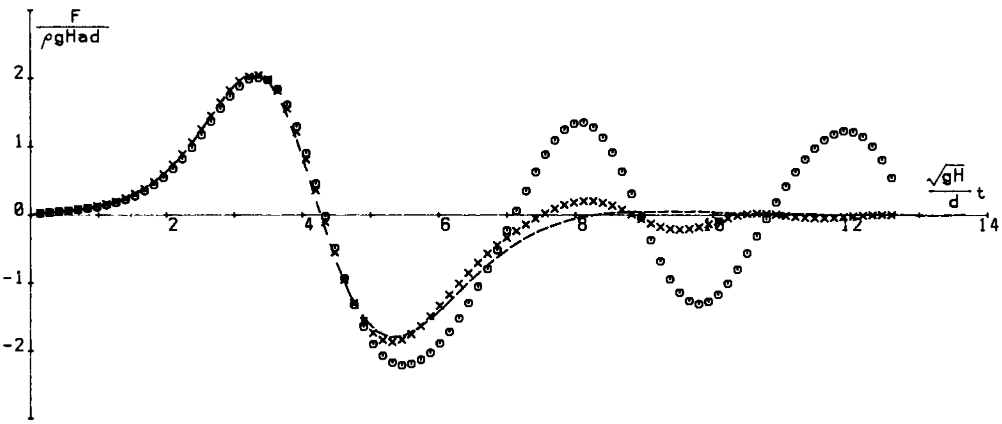

Figure 4. Wave force on a surface-piercing cylinder subjected to an incident solitary wave, with $d/R_1 = 0{\cdot}5$, $R_2 = 3{\cdot}0$, $H/d = 0{\cdot}1$

of relative domain size upon the wave forces exerted on the body. Comparisons of the results with the closed-form solution given by Isaacson[1] are presented in Figures 4–6. Taking advantage of symmetry, only one half of the cylinder has been considered. Along the x, y, ξ co-ordinates, grids are equally spaced, with node numbers m, n and l specified, respectively. In computation, a grid size of $m \times n \times l = 5 \times 15 \times 5$ was selected. The broken line indicates the closed-form solution given by Isaacson,[1] the points marked by '×' denote the numerical results by the present model, and the points marked by '○' indicate the numerical results incorporated with a prescribed incident wave form condition over the open boundary all the time. The last results are given for the case of $R_2 = 3$ only.

The program has been run on an STM-PC to generate available results without undue computer effort.

Prior to the analysis of the results, let us note a problem of the model outlined. Usually, the radiation condition over the open boundary is expressed in terms of the scattered wave. In the present model the diffraction wave has been identified as the difference between the total wave and the incident wave. This would be true for the linear wave diffraction problem. However, in the non-linear case, the outward travelling scattered wave will impact with the subsequent incident wave. Theoretically, there will be energy exchange, and the subsequent incident wave approaching the open boundary will experience some modification. Let us denote the incident wave approaching the open boundary without energy exchange as the ideal incident wave, say

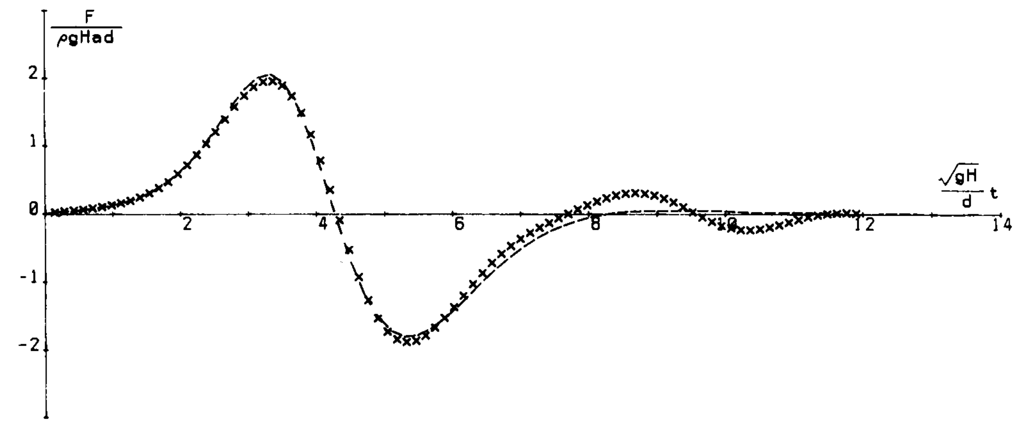

Figure 5. Wave force on a surface-piercing cylinder subjected to an incident solitary wave, with $d/R_1 = 0{\cdot}5$, $H/d = 0{\cdot}1$, $R_2 = 3{\cdot}5$

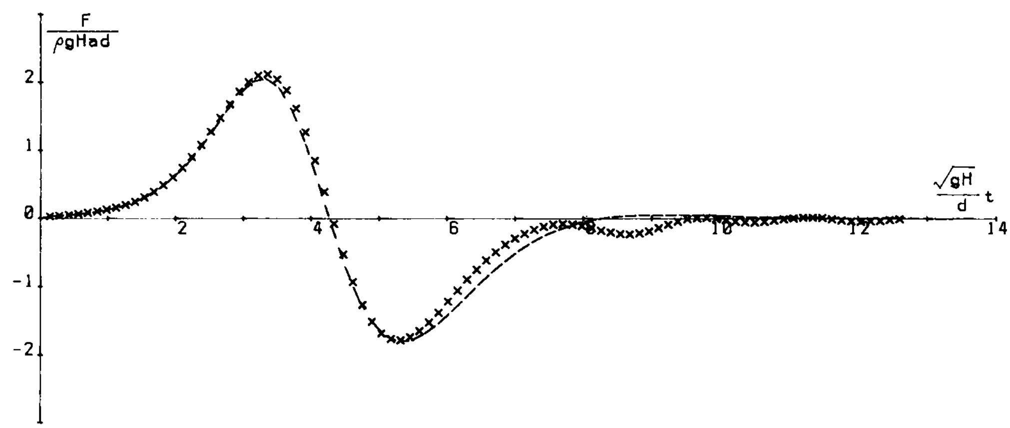

Figure 6. Wave force on a surface-piercing cylinder subjected to an incident solitary wave, with $d/R_1 = 0{\cdot}5$, $H/d = 0{\cdot}1$, $R_3 = 2{\cdot}5$

ϕ_{I}, and the one with energy exchange as real incident wave, say ϕ_{I}'. Now, two expressions for the scattered wave can be written as

$$\phi_s' = \phi - \phi_{\mathrm{I}}', \tag{48}$$

$$\phi_s = \phi - \phi_{\mathrm{I}}. \tag{49}$$

Because it is difficult to prescribe ϕ_s', we would rather use expression (49) in applications. Rewrite equation (49) as

$$\phi_s = (\phi - \phi_{\mathrm{I}}') + (\phi_{\mathrm{I}}' + \phi_{\mathrm{I}}) = \phi_s' + \Delta\phi_{\mathrm{I}}, \tag{50}$$

where $\Delta\phi_{\mathrm{I}}$ is the increment of the incident wave due to energy exchange before it approaches the open boundary. Obviously, the present model is correct, provided that $\Delta\phi_{\mathrm{I}}$ is a small quantity relative to ϕ_{I}'. Now, we shall split the open boundary into two portions, the up wave portion ACB and the off wave portion ADB, and discuss the generation of $\Delta\phi_{\mathrm{I}}$ and its effect upon these two portions. In part ACB, exterior to the open boundary, the incident wave has energy exchange with the scattered wave. Because the scattered wave there becomes very weak, the energy exchange will be small. In part ADB, interior to the open boundary, the incident wave impacts with the scattered wave and a much greater increment than that in the up wave portion is present. However, we cannot therefore conclude that the increment $\Delta\phi_{\text{I}}$ in the off wave portion will have a more seriously effect upon the computed results than that in the up wave portion. Practically, equations (48) and (49) can be considered as representing the total wave decomposed into incident and scattered components. According to equation (49), both incident wave and scattered wave have increments $\Delta\phi_{\text{I}}$. In the up wave portion, as the incident wave incorporating the increment $\Delta\phi_{\text{I}}$ approaches the computing region, there will be effects on the wave force calculation at subsequent times. In the off wave portion, the incident wave with increment and the scattered wave are propagating out of the computing region, so that there will be no serious effects on the computing wave force at subsequent times.

CONCLUSIONS

To sum up, the present model has considered sufficiently the nature of non-linearity interior to the computing domain, whereas exterior to it, especially over the up wave portion, the non-linear energy exchange between incident wave and scattered wave has been ignored. Experimental results are expected to provide further the extent of validity of the present model. Ideally, a theoretical estimation of the increment $\Delta\phi_{\text{I}}$ is preferable.

It can obviously be seen that in the model where a prescribed incident wave of permanent form is given over the open boundary, the results computed in the later period are significantly distorted. Results evaluated by the present method agree quite well with the closed-form solution, and it serves to highlight the fact that the open boundary condition presented in this paper is capable of effectively absorbing the scattered wave so that it does not reflect inwards, and also enables the computation of the wave force run over a sufficient duration for the wave motion to become fully established in the vicinity of the body.

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